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SOME PROBLEMS OF THE STABILITY OF CYLINDRICAL AND CONICAL SHELLS*

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The problem of the buckling of a membrane state of stress of a thin elastic shell is considered in a linear approximation. It is assumed that the buckling is accompanied by the formulation of a large number of dents. In the simplest case when the initial stresses and curvature of the middle surface are constant, the dents cover the whole shell surface /1–3/. If the quantities mentioned are not constant, the buckling pattern is complicated; localization of the dents can occur in the neighbourhoods of certain "weakest" lines /3–5/ or points /6/. The problem of the buckling of a shell of zero curvature is considered below. This is characterized, by the fact that the dents are stretched strongly along asymptotic lines and are localized near one (the weakest). The method is applicable to convex conical and cylindrical shells of medium length and not absolutely circular section; the shell edges are not necessarily plane curves. The two-dimensional problem reduces to a sequence of one-dimensional boundary value problems, while for a cylindrical shell, under certain particular assumptions, the approximate solution is obtained in closed form. A conical shell is considered, and the changes which must be made in the case of a cylindrical shell are outlined.

1. Let us introduce an orthogonal system of coordinates s, φ on the middle surface of a conical shell, where $s = s^\circ R^{-1}, s^\circ$ is the distance to the apex of the cone, R is the characteristic dimension of the middle surface, and φ is a coordinate on the directrix, selected in such a manner that the first quadratic form of the surface has the form $d\sigma^2 = R^2(ds^2 + s^2 d\varphi^2)$. Here the radius of curvature is $R_2 = Rsk^{-1}$. Let the shell be closed in the φ direction and bounded by two edges (φ_1 is the length of the curve formed when the cone and a sphere of radius R with centre at the apex of the cone intersect)

$$s_1(\varphi) \leqslant s \leqslant s_2(\varphi),\ 0 \leqslant \varphi \leqslant \varphi_1 \tag{1.1}$$

We will use the set of shallow-shell equations

$$\varepsilon^4\Delta^2 w + \lambda\Delta_T w + \Delta_k\Phi = 0,\ \ \varepsilon^4\Delta^2\Phi - \Delta_k w = 0 \tag{1.2}$$

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$$\Delta_T w = \frac{\varepsilon^2}{s^2}\left[\frac{\partial}{\partial\varphi}\left(T_2\frac{\partial w}{\partial\varphi}\right) + s\frac{\partial}{\partial s}\left(S\frac{\partial w}{\partial\varphi}\right) + s\frac{\partial}{\partial\varphi}\left(S\frac{\partial w}{\partial s}\right) + s\frac{\partial}{\partial s}\left(sT_1\frac{\partial w}{\partial s}\right)\right]$$

$$\Delta w = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial w}{\partial s}\right)\frac{\partial^2 w}{\partial s^2} + \frac{1}{s^2}\frac{\partial^2 w}{\partial\varphi^2}, \quad \Delta_k w = \frac{k}{s}\frac{\partial^2 w}{\partial s^2}, \quad w = R\varepsilon^4 w'$$

$$\Phi = \frac{\Phi'}{Eh}, \quad \varepsilon^8 = \frac{h^2}{12(1-\nu^2)R^2}$$

Here w' and Φ' are the normal deflection and the stress function, E, ν, h are Young's modulus, Poisson's ratio, and the shell thickness, and $\varepsilon > 0$ is a small parameter. The functions T_i, S are related to the initial forces T_i', S' by the formulas

$$T_i' = -Eh\lambda\varepsilon^6 T_i, \quad S' = -Eh\lambda\varepsilon^6 S \tag{1.3}$$

where $\lambda > 0$ is the desired loading parameter. The functions $k(\varphi) > 0$, $s_i(\varphi)$, $T_i(s, \varphi)$, $S(s, \varphi)$ are assumed to be infinitely differentiable.

2. The shell state of stress comprises the fundamental state of stress and a simple edge effect near the edges (1.1). Four homogeneous boundary conditions are given on the edges (1.1), but, only two conditions can be satisfied on each edge when constructing the fundamental state of stress. To formulate these conditions two such linear combinations of boundary conditions should be compiled in which the integrals of the edge effect do not occur, just as was done, for instance, in (7).

We will limit ourselves here to considering just one special case, namely, hinged clamping

$$T_n = v_l = w = M_n = 0 \quad \text{when} \quad s = s_1, \quad s = s_2 \tag{2.1}$$

where T_n, M_n are the force and momentum in a direction perpendicular to the edge, and v_l is the displacement in a direction tangent to the edge. The conditions

$$w = \Phi = 0 \quad \text{when} \quad s = s_1, \quad s = s_2 \tag{2.2}$$

should be satisfied to terms of the order of ε^2 in constructing the fundamental state of stress.

If the edge, a plane edge, is supported on a diaphragm that is rigid in its plane and flexible out of the plane, then the formulation of conditions (2.1) changes, while conditions (2.2) are conserved to the same accuracy.

3. We will seek a solution of system (1.2) in the form

$$w(s, \varphi, \varepsilon) = w_* \exp\{i\varepsilon^{-1}[q(\varphi-\varphi_0) + {}^1/_2 a(\varphi-\varphi_0)^2]\} \tag{3.1}$$

$$\lambda = \lambda_0 + \varepsilon\lambda_2 + \varepsilon^2\lambda_4 + \ldots$$

$$\left(w_* = \sum_{n=0}^{\infty}\varepsilon^{n/2}w_n(\xi, s), \xi = \varepsilon^{-1/2}(\varphi-\varphi_0), \operatorname{Im} a > 0\right)$$

where $w_n(\xi, s)$ are polynomials in ξ and the function Φ is sought in the same form (3.1). The number q is real and determines the variability in the φ direction, the generator $\varphi = \varphi_0$ is weakest, and the parameter a characterizes the rate of decrease of the dent depth with distance from it.

To determine the unknown functions w_n, Φ_n and the numbers q, a, φ_0, λ_n, we substitute (3.1) into (1.2) and equate the coefficients of powers of $\varepsilon^{1/2}$ to zero. It is convenient to express Φ_* first in terms of w_* by using the second equation in (1.2). We obtain

$$\Phi_* = \Delta_s\left[\frac{w_*}{q^4} - \frac{4\varepsilon^{1/2}}{q^5}\left(a\xi w_* - i\frac{\partial w_*}{\partial\xi}\right) + \frac{10\varepsilon}{q^6}\left(a^2\xi^2 w_* - 2ia\xi\frac{\partial w_*}{\partial\xi} - iaw_* - \frac{\partial^2 w_*}{\partial\xi^2}\right) + \ldots\right], \quad \Delta_s = s^3\frac{\partial^2}{\partial s^2} \tag{3.2}$$

Now the first equation in (1.2) yields a sequence of equations to determine w_n, which can be written in the form

$$H_0 w_0 = 0, \; H_0 w_1 + H_1 w_0 = 0, \; H_0 w_2 + H_1 w_1 + H_2 w_0 = 0, \; \ldots \tag{3.3}$$

Here

$$H_0 z = \frac{k^2\Delta_s^2}{s^3 q^4}z + \frac{q^4}{s^3}z + \lambda_0 N z, \qquad Nz = -\frac{q^2 T_2}{s}z \tag{3.4}$$

$$H_1 z = \left(a\frac{\partial H_0}{\partial q} + \frac{\partial H_0}{\partial\varphi}\right)\xi z - i\frac{\partial H_0}{\partial q}\frac{\partial z}{\partial\xi}, \qquad H_* z = i\lambda_0 q\left(\frac{\partial S}{\partial s}z + 2S\frac{\partial z}{\partial s}\right)$$

$$H_2 z = \frac{1}{2}\left(a^2\frac{\partial^2 H_0}{\partial q^2} + 2a\frac{\partial^2 H_0}{\partial q\,\partial\varphi} + \frac{\partial^2 H_0}{\partial\varphi^2}\right)\xi^2 z -$$

$$i\left(a\frac{\partial^2 H_0}{\partial q^2}+\frac{\partial^2 H_0}{\partial q\,\partial\varphi}\right)\xi\frac{\partial z}{\partial\xi}-\frac{1}{2}\frac{\partial^2 H_0}{\partial q^2}\left(iz+\frac{\partial^2 z}{\partial\xi^2}\right)-\frac{i}{2}\frac{\partial^2 H_0}{\partial q\,\partial\varphi}z+H_* z+\lambda_1 Nz$$

Substituting (3.1) into the condition $w=0$ for $s=s_k(\varphi)$ we obtain

$$w_0(\xi,s)=0,\quad w_1(\xi,s)+\xi s'\frac{\partial w_0}{\partial s}=0 \tag{3.5}$$

$$w_2(\xi,s)+\xi s'\frac{\partial w_1}{\partial s}+\frac{\xi^2}{2}\left((s')^2\frac{\partial^2 w_0}{\partial s^2}+s''\frac{\partial w_0}{\partial s}\right)=0,\ldots$$

and we have from the condition $\Phi=0$, taking (3.2) into account,

$$\frac{\partial^2 w_0}{\partial s^2}=0,\quad \frac{\partial^2 w_1}{\partial s^2}+\xi s'\frac{\partial^3 w_0}{\partial s^3}=0 \tag{3.6}$$

$$\frac{\partial^2 w_2}{\partial s^2}+\xi s'\frac{\partial^3 w_1}{\partial s^3}+\frac{\xi^2}{2}\left((s')^2\frac{\partial^4 w_0}{\partial s^4}+s''\frac{\partial^3 w_0}{\partial s^3}\right)-\frac{4is'}{q}\frac{\partial^2 w_0}{\partial s^2}=0,\ldots$$

4. Let us consider the boundary value problem originating in the zeroth approximation

$$H_0 w_0\equiv\frac{k^2(\varphi_0)}{q^4}\frac{\partial^2}{\partial s^2}\left(s^3\frac{\partial^2 w_0}{\partial s^2}\right)+\frac{q^4}{s^3}w_0+\lambda_0 N w_0=0 \tag{4.1}$$

$$w_0=\frac{\partial^2 w_0}{\partial s^2}=0 \text{ when } s=s_1(\varphi_0),\quad s=s_2(\varphi_0)$$

In addition to the fundamental parameter λ_0 this problem still contains two parameters, q and φ_0, and the least eigenvalue will be

$$\lambda_0^\circ=\min_{q,\varphi_0}\lambda_0(q,\varphi_0)=\lambda_0(q^0,\varphi_0^0)$$

The following relationships should be satisfied

$$\frac{\partial\lambda_0}{\partial q}=\frac{\partial\lambda_0}{\partial\varphi_0}=0 \text{ when } q=q^0,\ \varphi_0=\varphi_0^0\ (\lambda_0=\lambda_0^0) \tag{4.2}$$

Now, let

$$\Lambda=\begin{Vmatrix}\lambda_{qq} & \lambda_{q\varphi}\\ \lambda_{q\varphi} & \lambda_{\varphi\varphi}\end{Vmatrix} \tag{4.3}$$

denote a matrix comprised of the second derivatives of the function $\lambda_0(q,\varphi_0)$ for $q=q^\circ$, $\varphi_0=\varphi_0^\circ$. We assume that Λ is a positive-definite matrix.

To calculate the derivatives occurring in (4.2) and (4.3), problem (4.1) can be differentiated with respect to the parameters q and φ_0. For instance

$$H_0 w_q+\frac{\partial H_0}{\partial q}w_0+\frac{\partial\lambda_0}{\partial q}Nw_0=0,\quad w_q=\frac{\partial^2 w_q}{\partial s^2}=0 \tag{4.4}$$

$$H_0 w_\varphi+\frac{\partial H_0}{\partial\varphi_0}w_0+\frac{\partial\lambda_0}{\partial\varphi_0}Nw_0=0$$

$$w_\varphi+s'\frac{\partial w_0}{\partial s}=\frac{\partial^2 w_\varphi}{\partial s^2}+s'\frac{\partial^3 w_0}{\partial s^3}=0,\quad s=s_1,\quad s=s_2$$

Under the conditions (4.2), problems (4.4) will be inhomogeneous problems "in the spectrum". The condition for a solution of the problem in the spectrum to exist

$$H_0 z+G(s)=0,\quad z+g_{0k}=\frac{\partial^2 z}{\partial s^2}+g_{2k}=0 \text{ for } s=s_k \tag{4.5}$$

is the equation

$$\int_{s_1}^{s_2}\bar w_0 G\,ds+\frac{k^2 s^3}{q^4}\left(g_{0k}\frac{\partial^3\bar w_0}{\partial s^3}+g_{2k}\frac{\partial\bar w_0}{\partial s}\right)\Big|_{s_1}^{s_2}=0 \tag{4.6}$$

Now Eqs.(4.2) become

$$\int_{s_1}^{s_2}\bar w_0\frac{\partial H_0}{\partial q}w_0\,ds=0 \tag{4.7}$$

$$\int_{s_1}^{s_2}\bar w_0\frac{\partial H_0}{\partial\varphi_0}w_0\,ds+\frac{k^2 s^3 s'}{q^4}\left(\frac{\partial w_0}{\partial s}\frac{\partial^3\bar w_0}{\partial s^3}+\frac{\partial\bar w_0}{\partial s}\frac{\partial^3 w_0}{\partial s^3}\right)\Big|_{s_1}^{s_2}=0$$

Of the elements of the matrix (4.3) we will write only the expression for λ_{qq}

$$\lambda_{qq}\int_{s_1}^{s_2}\bar w_0 N w_0\,ds+\int_{s_1}^{s_2}\bar w_0\left(2\frac{\partial H_0}{\partial q}w_q+\frac{\partial^2 H_0}{\partial q^2}w_0\right)ds=0 \tag{4.8}$$

5. We will now solve the sequence of equations (3.3), taking the boundary conditions

(3.5) and (3.6) into account. In a zeroth approximation we obtain

$$w_0(\xi, s) = P_0(\xi)\, w_0^\circ(s) \tag{5.1}$$

where w_0° is the eigenfunction of problem (4.1), under conditions (4.2), and P_0 is a function still not determined. To a first approximation

$$H_0 w_1 + \left[\xi P_0\left(a\frac{\partial H_0}{\partial q} + \frac{\partial H_0}{\partial \varphi_0}\right) - iP_0'\frac{\partial H_0}{\partial q}\right] w_0^0 = 0 \tag{5.2}$$
$$w_1 + \xi P_0 s'\frac{\partial w_0^0}{\partial s} = \frac{\partial^2 w_1}{\partial s^2} + \xi P_0 s'\frac{\partial^2 w_0^0}{\partial s^2} = 0 \quad \text{for} \quad s = s_k$$

Because of (4.4)-(4.7), the condition for a solution to problem (5.2) to exist is equivalent to the Eqs.(4.2), from which the parameters q^0 and φ_0^0 were determined. Hence, in particular, it follows that not just any generatrix φ_0 can be taken as the weakest in (3.1). We find from (5.2)

$$w_1(\xi, s) = P_1(\xi)\, w_0^0 + \xi P_0(aw_q + w_\varphi) - iP_0' w_q \tag{5.3}$$

where w_q, w_φ are solutions of problems (4.4) for $w_0 = w_0^0$, and the function P_1 is still not determined.

To a second approximation taking account of the boundary conditions (3.5),(3.6), we have the following equation from which we obtain, by virtue of (4.6) the condition for solution w_2 to exist

$$LP_0 \equiv -\tfrac{1}{2}\lambda_{qq}P_0'' + b\xi P_0' + (\eta - \lambda_2 + \tfrac{1}{2}b + c\xi^2)\, P_0 = 0 \tag{5.4}$$
$$b = -i\,(a\lambda_{qq} + \lambda_{q\varphi}), \quad 2c = a^2\lambda_{qq} + 2a\lambda_{q\varphi} + \lambda_{\varphi\varphi}$$

$$\eta = \frac{i}{2z}\left\{\int_{s_1}^{s_2}\left(w_0^0\frac{\partial \overline{H}_0}{\partial q}\overline{w}_\varphi - \overline{w}_0^0\frac{\partial H_0}{\partial q}w_\varphi\right)ds - 2i\int_{s_1}^{s_2}\overline{w}_0^0 H_* w_0^0\, ds + \right.$$
$$\left.\frac{4k^2s^2s'}{q^5}\left(\frac{dw_0^0}{ds}\frac{d^3\overline{w}_0^0}{ds^3} - \frac{d\overline{w}_0^0}{ds}\frac{d^3w_0^0}{ds^3}\right)\Bigg|_{s_1}^{s_2}\right\}, \quad z = -\int_{s_1}^{s_2}\overline{w}_0^0 N w_0^0\, ds$$

The condition $c = 0$ is necessary for a solution of (5.4) to exist in the form of a polynomial. From the quadratic equation $c = 0$ we find a unique quantity a, since the matrix Λ is positive definite, such that $\operatorname{Im} a > 0$. For

$$c = 0, \quad \lambda_2 = (n + \tfrac{1}{2})\, b + \eta, \quad n = 0, 1, 2, \ldots \tag{5.5}$$

equation (5.4) has the solution $P_0 = H_n(\xi)$, where H_n is a Hermite polynomial of degree n. The case $n = 0, H_n \equiv 1$ is primarily of interest because the quantity λ_2 here is minimal (note that $b > 0$).

Subsequent approximations are constructed analogously. We simply note that $w_k(\xi, s)$ are even or odd polynomials of ξ while the condition for w_{k+2} to exist yields

$$LP_k + \lambda_k P_0 + F_k(\xi) = 0, \; k > 0 \tag{5.6}$$

where L is the operator on the left side of (5.4) for $c = 0$. The quantity λ_k is determined from the condition for the solution (5.6) to exist as a polynomial. The evenness of the polynomials w_k, P_k, F_k changes in proceeding to the next approximation; consequently $\lambda_k = 0$ for odd k, as noted in (3.1).

Separating the real and imaginary parts in (3.1), we obtain that each eigenvalue (3.1) is asymptotically double. The shape of the deflection has the form

$$w = (\operatorname{Re} w_* \cos z - \operatorname{Im} w_* \sin z)\exp\{-\tfrac{1}{2}\operatorname{Im} a\xi^2\} \tag{5.7}$$
$$z = \varepsilon^{-1/4}q^0\xi + \tfrac{1}{2}\operatorname{Re} a\xi^2 + \theta$$

The method utilized here does not enable the initial phase $\theta = \text{const}$ to be determined; it takes one of the two values $0 \leqslant \theta_1, \theta_2 < 2\pi$, and also does not enable the corresponding eigenvalues to be distinguished.

If follows from the method of constructing the solution that the requirement formulated above for the shell to be closed in the circumferential direction is not essential. It is just necessary that the weakest generatrix be sufficiently remote from the rectilinear edges.

6. Problem (4.1) has the eigenvalues $\lambda_0 > 0$ if $T_2(s, \varphi) > 0$ (which corresponds to compressive forces) on at least part of the shell. Let this condition be satisfied, and let S and T_1 have the same order of magnitude as T_2. Here the function w_0° is real and $\eta = 0$. It therefore follows that the forces S and T_1 are neither in the zeroth nor the first approximation of the loading parameter λ but only in the quantity λ_4 (see (3.1)), to calculate which two more approximations must be constructed.

To estimate the influence of S and T_1 on λ as well as to consider the case $T_2 \leqslant 0$ on the whole shell (the internal pressure), we temporarily assume that the quantities $T_2, \varepsilon S$ and εT_1 are of identical order. This causes a change in the operators N and H_* presented in (3.4)

$$N(z) = -\frac{q^2T_2}{s}z + i\varepsilon q\left(\frac{\partial S}{\partial s}z + 2S\frac{\partial z}{\partial s}\right) \tag{6.1}$$

$$H_* z = \varepsilon \left[\frac{\partial}{\partial s}\left(sT_1 \frac{\partial z}{\partial s}\right) - \frac{1}{2} \frac{\partial^2 S}{\partial s\, \partial \varphi_0} z \right]$$

while preserving the remaining formulas presented above. For such N the boundary value problem (4.1) also has a real spectrum, but the eigenfunctions w_0 are complex. The quantity λ_0 depends on the force S, but the force T_1 occurs in the quantities η and λ_2.

The case $T_2 \equiv 0$, $S \not\equiv 0$ is not excluded from consideration, but the case $T_2 = S \equiv 0$, $T_1 \not\equiv 0$ is not considered because the solution has a form different from (3.1).

7. We will now consider a cylindrical shell. Let its dimensions and shape be defined by the relations

$$s_1(\varphi) \leqslant s \leqslant s_2(\varphi), \quad 0 \leqslant \varphi \leqslant \varphi_2, \quad do^2 = R^2(ds^2 + d\varphi^2), \quad R_2 = \frac{R}{k(\varphi)} \tag{7.1}$$

where $\varphi_2 = 2\pi$ and $k(\varphi) \equiv 1$ for a circular cylindrical shell of radius R.

The direct passage from a conical to a cylindrical shell is not so simple because $s_1, s_2 \to \infty$ and $\varphi_1 \to 0$ in (1.1). However, all the formulas presented above can be utilized by replacing the factor s, which occurs there explicitly, by one.

For a cylindrical shell the boundary value problem (4.1) has an explicit solution (unlike the case of a conical shell) for $Nz = -q^2 T_2 z$ if T_2 is independent of s:

$$w_0 = \sin \frac{\pi(s - s_1(\varphi_0))}{l(\varphi_0)}, \quad l = s_2 - s_1, \quad \lambda_0 = \frac{q^2}{T_2} + \frac{k^2\pi^4}{T_2 l^4 q^6} \tag{7.2}$$

Hence we find

$$\lambda_0^0 = \min_{\varphi_0} \frac{4\pi k^{1/2}(\varphi_0)}{3^{3/4} l(\varphi_0) T_2(\varphi_0)}, \quad (q^0)^8 = \frac{3\pi^4 k^2(\varphi_0^0)}{l^4(\varphi_0^0)} \tag{7.3}$$

and the quantity λ_2 in (3.1) is found from (5.5) for $n = \eta = 0$ after the matrix Λ has been calculated by differentiating λ_0.

Note that the zeroth approximation of the critical load λ_0^0 agrees with the value determined by the Southwell-Papkovich formula (/3/, p.138) for a circular cylindrical shell of constant length $Rl(\varphi_0^0)$ and radius $R[k(\varphi_0^0)]^{-1}$.

8. To estimate the error of the method proposed we will select a problem for which a numerical solution is known (/3/, p.131). We will examine the stability of a cantilever circular cylindrical shell of constant length $L = lR$ subjected to an external pressure that is non-uniform over the circumference. In the notation used above

$$T_2' = pRT_2, \quad T_2 = 1 + \alpha_1 \cos\varphi + \alpha_2 \cos 2\varphi$$
$$S = s(\alpha_1 \sin\varphi + 2\alpha_2 \sin 2\varphi), \quad T_1 = -{}^1/_2 s^2 (\alpha_1 \cos\varphi + 4\alpha_2 \cos 2\varphi) \tag{8.1}$$

Let $\alpha_1 \geqslant 0, \alpha_2 \geqslant 0$. Then the generatrix $\varphi_0^\circ = 0$ will be the weakest. Retaining two terms in (3.1), we can write the formula for the critical value of the parameter p in the form

$$p = \frac{4\pi}{3^{3/4}} \frac{Eh\varepsilon^4 k_*}{L(1 + \alpha_1 + \alpha_2)}, \quad k_* = 1 + \frac{\varepsilon\lambda_2}{\lambda_0^0} \tag{8.2}$$

Calculations by the formulas in Sec. 5 yield

$$a = \frac{i}{4}[\lambda_0^0(\alpha_1 + 4\alpha_2)]^{1/2}, \quad \lambda_0^0 = \frac{4\pi 3^{-3/4}}{l(1 + \alpha_1 + \alpha_2)}, \quad \lambda_2 = \frac{-8ai}{1 + \alpha_1 + \alpha_2} \tag{8.3}$$

and the expression for the parameter k_* which takes account of the inhomogeneity of the loading becomes

$$k_* = 1 + 0.624\left(\frac{\alpha_1 + 4\alpha_2}{1 + \alpha_1 + \alpha_2}\right)^{1/2} y, \quad y = \left(\frac{h^2 L^4}{(1 - \nu^2) R^6}\right)^{1/8} \tag{8.4}$$

We will introduce again the half-wavelength of the dents in the circumferential direction $\Delta\varphi$ and the parameter ρ which takes account of the decrease in the dent depth with distance from the weakest generatrix. We obtain

$$\Delta\varphi = \frac{\pi\varepsilon}{q^0} = 1.132y, \quad \rho = \frac{(\Delta\varphi)^2 \operatorname{Im} a}{2\varepsilon} = 0.513\left(\frac{\alpha_1 + 4\alpha_2}{1 + \alpha_1 + \alpha_2}\right)^{1/2} y \tag{8.5}$$

Here the dent depths are proportional to the numbers $1, e^{-\rho}, e^{-4\rho}$, etc.

Utilization of (3.1) is based on the assumption of a large number of dents in the circumferential direction. Taking account of the expressions for $\Delta\varphi$ and y, we conclude that the method proposed here is suitable for medium length shells. As L increases, the quantity $\Delta\varphi$ grows and the accuracy of the method is reduced.

Qualitatively (8.4) agrees completely with the results presented in /3/. In quantitave respects, the error in (8.4) depends substantially on the parameters of the problem. The least favourable combination is $R/h = 100$, $L/R = 10$ for which the error reaches 15% for certain values of α_1 and α_2. As h/R and L/R decrease the error decreases and does not exceed 5% for $R/h = 400$ and $L/R = 2.5$.

The buckling mode is shown in Fig. 17.4 in /3/ for $R/h = 1000$, $L/R = 10$, $\alpha_1 = 0.5$ and $\alpha_2 = 0$. Dent localization in the neighbourhood of the weakest generatrix $\varphi_0^\circ = 0$ is clearly seen. In this case, formulas (8.5) yield $\Delta\varphi = 36.4°$ and $\rho = 0.186$, meaning that the dent amplitudes are proportional to the numbers 1, 0.83, 0.48, 0.19, 0.05. It is seen from Fig. 17.4 in /3/ that $\Delta\varphi = 30°$. Note that in this case the error in (8.4) is less than 1%.

Formula (8.4) does not take account of the initial forces T_1', S'. Their contribution can be estimated by means of (6.1) and (5.4). Consequently, the term $0.181\,(\alpha_1 + 4\alpha_2)(1 + \alpha_1 + \alpha_2)^{-1}y^2$, which takes account of terms of order ε^2 in k_*, is appended to the quantity k_*.

Even functions of φ were taken as coordinate functions of the Bubnov method in /3/. Hence, the buckling mode constructed $w(s, \varphi)$ is also an even function of φ. Buckling in an odd mode is also possible for a load very close to critical.

9. Let us consider the problem of the buckling of a circular cylindrical medium-length shell with an oblique edge subjected to a homogeneous external pressure p. Let

$$s_1(\varphi) = 0, \quad s_2(\varphi) = l(\varphi) = l_* + \operatorname{tg}\beta\cos\varphi \tag{9.1}$$

where β is the slope of the edge. It is necessary to assume $T_2 = 1$, $k = 1$ in the formulas in Sec. 7. Then

$$\varphi_0{}^0 = 0, \quad \lambda_0{}^0 = \frac{4\pi}{3^{3/4} l_0}, \quad (q^0)^8 = \frac{3\pi^4}{l_0^4}, \quad l_0 = l_* + \operatorname{tg}\beta$$
$$\lambda_{qq} = 16, \quad \lambda_{q\varphi} = 0, \quad \lambda_{\varphi\varphi} = \frac{4\pi\operatorname{tg}\beta}{3^{3/4} l_0^2} \tag{9.2}$$

and the critical value of the pressure is

$$p = \frac{4\pi}{3^{3/4}}\frac{Eh\varepsilon^6 k_*}{Rl_0}, \quad k_* = 1 + 0.624\left(\frac{h^2\operatorname{tg}^4\beta}{(1-\nu^2)R^2}\right)^{1/4} \tag{9.3}$$

For $k_* = 1$ we obtain the Southwell-Papkovich formula corresponding to the maximal shell length, while the second term in k_* takes account of the influence of the oblique edge. For instance, for $\beta = 45°$, $R/h = 400$, and $\nu = 0.3$ the presence of the oblique edge increases the critical pressure by 14%.

10. The method described can also be utilized to construct the lower part of the spectrum for the free vibrations of a conical shell and a medium-length cylindrical shell. For a conical shell we must put

$$\lambda\Delta_T w = -\lambda w, \quad \lambda = \frac{E\omega^2}{\rho R^2\varepsilon^4}, \quad Nz = -sz, \quad H_* z = 0 \tag{10.1}$$

in (1.2) and (3.4), where ω, ρ are the vibration frequency and material density. Asymptotically the double eigenvalues of the frequency parameter λ are given by

$$\lambda^{(n)} = \lambda_0{}^0 + \varepsilon\,(n + 1/2)\,b + O(\varepsilon^2), \quad n = 0, 1, 2, \ldots \tag{10.2}$$

where λ_0^0 is the eigenvalue of problem (4.1) while b is given by (5.4).

We change to a cylindrical shell (non-circular and with an oblique edge in the general case) exactly as in Sec.7. We have

$$\lambda_0 = q^4 + \pi^4 q^{-4} F^2(\varphi_0), \quad F = kl^{-2} \tag{10.3}$$

The function F takes the least value on the weakest generatrix $\varphi_0{}^0$. Evaluating the derivatives λ_{qq}, $\lambda_{\varphi\varphi}$ $(\lambda_{q\varphi} = 0)$, we find from (10.2) and (5.4)

$$\lambda^{(n)} = 2\pi^2 F_0\,[1 + 4\,(n + 1/2)\varepsilon\pi^{-1/2}(F_0'')^{1/2}] + O(\varepsilon^2)\,F_0^{-3/4} \tag{10.4}$$

where $F_0 = F(\varphi_0{}^0)$. Hence, the formula for a non-circular cylindrical shell of constant length (/8/, p.335) is obtained as a special case when $l = \text{const}$, $n = 0$. Note that there is a misprint in formulas (6.23) and (6.24) in /8/: the coefficient 8 should be replaced by 2.

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ACOUSTIC WAVE INTERACTION WITH BODIES COVERED BY A THIN COMPRESSIBLE LAYER*

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The problem of acoustic wave interaction with rigid bodies on whose surface there is a thin compressible layer is formulated. The motion of the material is assumed to be quasi-two-dimensional in the layer, which results in a problem with special boundary condition, which generalizes the problem of acoustic wave diffraction by a rigid body and a cavity. The problem of plane acoustic wave diffraction by a sphere covered with a thin compressible layer is solved.

1. Formulation of the Problem.

Let curvilinear orthogonal coordinates in space define the radius-vector of the point $\mathbf{r}(q_1, q_2, q_3)$. A thin layer of an ideal compressible fluid of variable thickness, whose outer surface is $\mathbf{r} = \mathbf{r}(q_1, q_2, a+h)$ where h is a function of q_1, q_2 and time t, is attached to the surface of the rigid body described by the parametric equation $\mathbf{r} = \mathbf{r}(q_1, q_2, a)$, where a is a constant. The space outside the body and the compressible layer is filled with an ideal fluid with physical characteristics different from the characteristics of the layer material on the body. The reflection of a pressure wave from the body is investigated later. The problem under consideration arises when studying the diffraction by bodies with thin damping coatings that are in water when there are gas bubbles on their surface, and in other cases.

In general, when a pressure wave acts on a body covered by a compressible layer, complex three-dimensional fluid flow occurs in the layer. Because of the thinness of the layer, it is natural to try to reduce the problem of the flow in a layer to a quasi-two-dimensional flow over a surface $\mathbf{r} = \mathbf{r}(q_1, q_2, a)$ /1/. Let us formulate the constraints on the conditions of the problem under which this can be done successfully. First, because of the thinness of the layer, the Lamé parameters $H_1 = |\partial\mathbf{r}/\partial q_1|$ and $H_2 = |\partial\mathbf{r}/\partial q_2|$ can be assumed to be independent of the coordinate q_3 for $a \leqslant q_3 \leqslant a+h$. It can be shown that this assumption will be satisfied with sufficient accuracy, when appropriate derivatives of $\mathbf{r}(q_1, q_2, q_3)$ exist, if the following inequalities are satisfied

$$\left|\frac{\partial H_i(q_1, q_2, a)}{\partial q_3}\right| \frac{h(q_1, q_2, t)}{H_i(q_1, q_2, a)} \ll 1, \quad i = 1, 2 \tag{1.1}$$

To simplify the calculations, the coordinate q_3 is identified with the arc-length of the appropriate coordinate line, i.e., it is assumed that $H_3 = |\partial\mathbf{r}/\partial q_3| = 1$, as can always be done by an appropriate selection of the coordinate system.

We will further assume that the pressure p in the layer and the density ρ are independent of the coordinate q_3. Introducing this assumption, we will neglect the waves in the direction of the normal to the surface $\mathbf{r} = \mathbf{r}(q_1, q_2, a)$. For the one-dimensional case, the validity of this assumption of proved in /1/ in an acoustic formulation for a small value of the ratio between the acoustic impedance of the layer material and the acoustic impedance of the surrounding fluid. Moreover, it is assumed, for simplicity, that the flows in the layer and in the surrounding fluid are barotropic.

Let v_1, v_2, v_3 be components of the fluid velocity vector in the layer. It is assumed that the components v_1 and v_2 depend slightly on the coordinate q_3, and we also consider their mean values

$$\langle v_i \rangle = \frac{1}{h} \int_a^{a+h} v_i \, dq_3 \quad (i = 1, 2)$$

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